On the power of normal form theory:

Simplification of a Hamiltonian dynamical system using normal form theory and quasiintegrals of motion

by Martin Engel

Consider the well-known *Størmer problem* which is given by the Hamiltonian

$$H(\rho, z, p_\rho, p_z) \quad = \quad \frac{1}{2}\left(p_\rho^2 + p_z^2\right) + \frac{1}{2}\left(\frac{1}{\rho} - \frac{\rho}{r^3}\right)^2 \qquad \left(\text{where} \quad r = \sqrt{\rho^2 + z^2}\right),$$

completely describing the classical motion of a charged particle in the earth's magnetic field (which is assumed here to be a pure dipole field).

As the first step of our simplification procedure, we transform from the ordinary cylindrical polar coordinates $(\rho, \not\varphi, z)$ used above to *dipolar coordinates* (a, b) and expand the resulting expression in terms of these new coordinates and their corresponding canonically conjugate momenta p_a, p_b (cf. G. Contopoulos, L. Vlahos, J. Math. Phys. **16** (1975) 1469–1474):

$$\begin{aligned} H(a,b,p_a,p_b) \quad = \quad & 0.5p_b^2 + 0.5p_a^2 + 0.5a^2 - 3ap_b^2 - 2a^3 + 10.5a^2p_b^2 + 5a^4 + 3b^2p_a^2 + 4.5b^2p_b^2 + 1.5b^2a^2 - 28a^3p_b^2 - 10a^5 \\ & + 12b^2ap_a^2 - 9b^2ap_b^2 + 63a^4p_b^2 + 17.5a^6 + 18b^2a^2p_a^2 + 13.5b^2a^2p_b^2 - 4.5b^4p_a^2 + 1.5b^4p_b^2 - 3b^4a^2 - 126a^5p_b^2 \\ & - 28a^7 + 12b^2a^3p_a^2 - 18b^2a^3p_b^2 - 36b^4ap_a^2 + 3b^4ap_b^2 - 12b^4a^3 + 231a^6p_b^2 + 42a^8 + 3b^2a^4p_a^2 + 22.5b^2a^4p_b^2 \\ & - 126b^4a^2p_a^2 + 1.5b^4a^2p_b^2 - 18b^4a^4 + 20b^6p_a^2 + 2.5b^6p_b^2 + 14b^6a^2 - 396a^7p_b^2 - 60a^9 - 27b^2a^5p_b^2 - 252b^4a^3p_a^2 \\ & - 12b^4a^5 + 240b^6ap_a^2 + 15b^6ap_b^2 + 112b^6a^3 + 643.5a^8p_b^2 + 82.5a^{10} + 31.5b^2a^6p_b^2 - 315b^4a^4p_a^2 - 3b^4a^6 \\ & + 1320b^6a^2p_a^2 + 37.5b^6a^2p_b^2 + 392b^6a^4 - 115.5b^8p_a^2 - 27b^8p_b^2 - 82.5b^8a^2 - 1001a^9p_b^2 - 110a^{11} - 36b^2a^7p_b^2 \\ & - 252b^4a^5p_a^2 + 4400b^6a^3p_a^2 + 50b^6a^3p_b^2 + 784b^6a^5 - 1848b^8ap_a^2 - 270b^8ap_b^2 - 990b^8a^3 + 1501.5a^{10}p_b^2 + 143a^{12} \\ & + 40.5b^2a^8p_b^2 - 126b^4a^6p_a^2 + 9900b^6a^4p_a^2 + 37.5b^6a^4p_b^2 + 980b^6a^6 - 13860b^8a^2p_a^2 - 1215b^8a^2p_b^2 - 5445b^8a^4 \\ & + 756b^{10}p_a^2 + 214.5b^{10}p_b^2 + 546b^{10}a^2 + \mathcal{O}(13) \end{aligned}$$

Now *generalized normal form theory* (U. M. Engel, B. Stegemerten, P. Eckelt, J. Phys. A: Math. Gen. **28** (1995) 1425–1448) can be applied to this transformed Hamiltonian, yielding the following most simple normal form for the Størmer problem:

$$\begin{aligned} G(a,b,p_a,p_b) \quad = \quad & 0.5p_b^2 + 0.5p_a^2 + 0.5a^2 - 1.875p_a^4 - 3.75a^2p_a^2 - 1.875a^4 + 2.25b^2p_a^2 + 2.25b^2a^2 + 0.9375p_a^6 \\ & + 2.8125a^2p_a^4 + 2.8125a^4p_a^2 + 0.9375a^6 + 0.14062b^2p_a^4 + 0.28125b^2a^2p_a^2 + 0.14062b^2a^4 + 2.4375b^4p_a^2 \\ & + 2.4375b^4a^2 - 3.04687p_a^8 - 12.1875a^2p_a^6 - 18.2812a^4p_a^4 - 12.1875a^6p_a^2 - 3.04687a^8 + 371.125b^2p_a^6 \\ & + 1113.375b^2a^2p_a^4 + 1113.375b^2a^4p_a^2 + 371.125b^2a^6 - 148.5b^4p_a^4 - 297b^4a^2p_a^2 - 148.5b^4a^4 + 4.2312b^6p_a^2 \\ & + 4.2312b^6a^2 - 15.2578p_a^{10} - 76.2890a^2p_a^8 - 152.578a^4p_a^6 - 152.578a^6p_a^4 - 76.289a^8p_a^2 - 15.257a^{10} \\ & + 10961b^2p_a^8 + 43844.1b^2a^2p_a^6 + 65766.2b^2a^4p_a^4 + 43844.1b^2a^6p_a^2 + 10961b^2a^8 - 5761.5b^4p_a^6 - 17284.6b^4a^2p_a^4 \\ & - 17284.6b^4a^4p_a^2 - 5761.5b^4a^6 + 225.87b^6p_a^4 + 451.74b^6a^2p_a^2 + 225.87b^6a^4 - 2.16b^8p_a^2 - 2.160b^8a^2 \\ & - 96.202p_a^{12} - 577.212a^2p_a^{10} - 1443.03a^4p_a^8 - 1924.04a^6p_a^6 - 1443.03a^8p_a^4 - 577.21a^{10}p_a^2 - 96.202a^{12} \\ & + 362429.8b^2p_a^{10} + 1812149.4b^2a^2p_a^8 + 3624298.8b^2a^4p_a^6 + 3624298.8b^2a^6p_a^4 + 1812149.4b^2a^8p_a^2 + 362429.8b^2a^{10} \\ & - 310044.3b^4p_a^8 - 1240177.5b^4a^2p_a^6 - 1860266.3b^4a^4p_a^4 - 1240177.5b^4a^6p_a^2 - 310044.3b^4a^8 + 62380.4b^6p_a^6 \\ & + 187141.4b^6a^2p_a^4 + 187141.4b^6a^4p_a^2 + 62380.4b^6a^6 - 2779.1b^8p_a^4 - 5558.2b^8a^2p_a^2 - 2779.1b^8a^4 + 18.898b^{10}p_a^2 \\ & + 18.898b^{10}a^2 + \mathcal{O}(13) \end{aligned}$$

The third, and final, step on our path towards maximum simplicity is the transformation of the above normal form into the corresponding quasiintegral of motion; for our little sample problem, a minute's thought will reveal that we have:

$$\begin{aligned} I(a,b,p_a,p_b) \quad = \quad & 0.5p_a^2 + 0.5a^2 - 3ap_b^2 - 2a^3 + 1.875p_a^4 + 3.375p_b^2p_a^2 + 4.5p_b^4 + 3.75a^2p_a^2 + 14.625a^2p_b^2 + 6.875a^4 \\ & - 1.5bap_bp_a + 0.75b^2p_a^2 - 0.75b^2a^2 - 63.5ap_b^2p_a^2 - 78.75ap_b^4 - 15a^3p_a^2 - 106.3333333a^3p_b^2 - 25a^5 + 41bp_bp_a^3 \\ & + 58.5bp_b^3p_a + 36ba^2p_bp_a + 12b^2ap_a^2 + 4.5b^2ap_b^2 + 9b^2a^3 + 13.125p_a^6 - 3.513671875p_b^2p_a^4 + 97.828125p_b^4p_a^2 \\ & + 158.625p_b^6 + 39.375a^2p_a^4 + 156.3320313a^2p_b^2p_a^2 + 613.453125a^2p_b^4 + 76.875a^4p_a^2 + 512.1113281a^4p_b^2 + 98.125a^6 \\ & + 76.78125bap_bp_a^3 - 157.5bap_b^3p_a - 91.78125ba^3p_bp_a - 18.140625b^2p_a^4 - 21.375b^2p_b^2p_a^2 - 6.75b^2p_b^4 \\ & - 34.03125b^2a^2p_a^2 - 84.375b^2a^2p_b^2 - 55.265625b^2a^4 + 9.75b^3ap_bp_a - 3.5625b^4p_a^2 + 4.6875b^4a^2 - 1648.4125ap_b^2p_a^4 \\ & - 6817.992188ap_b^4p_a^2 - 7089.46875ap_b^6 - 157.5a^3p_a^4 - 3599.232292a^3p_b^2p_a^2 - 9444.148438a^3p_b^4 \\ & - 390a^5p_a^2 - 3433.927292a^5p_b^2 - 410.5a^7 + 1412.1625bp_bp_a^5 + 6860.15625bp_b^3p_a^3 \\ & + 6502.5bp_b^5p_a + 2785.5625ba^2p_bp_a^3 + 7177.03125ba^2p_b^3p_a + 1475.75ba^4p_bp_a + 336b^2ap_a^4 \\ & + 2569.6875b^2ap_b^2p_a^2 + 2481.75b^2ap_b^4 + 766.125b^2a^3p_a^2 + 2850.1875b^2a^3p_b^2 + 529.125b^2a^5 \\ & - 301.5b^3p_bp_a^3 - 296.25b^3p_b^3p_a - 351b^3a^2p_bp_a - 90b^4ap_a^2 - 43.125b^4ap_b^2 \\ & - 69.75b^4a^3 + 117.3046875p_a^8 - 2570.914545p_b^2p_a^6 - 8303.522835p_b^4p_a^4 + 8629.382812p_b^6p_a^2 \\ & + 19073.95312p_b^8 + 469.21875a^2p_a^6 - 4049.921175a^2p_b^2p_a^4 + 20162.08167a^2p_b^4p_a^2 + 71010.21094a^2p_b^6 \\ & + 1097.578125a^4p_a^4 + 13346.76183a^4p_b^2p_a^2 + 64281.06506a^4p_b^4 + 2017.96875a^6p_a^2 + 19344.57107a^6p_b^2 \\ & + 1801.804688a^8 + 2801.080078bap_bp_a^5 - 12891.51562bap_b^3p_a^3 - 40729.92188bap_b^5p_a - 5821.457031ba^3p_bp_a^3 \\ & - 40895.03906ba^3p_b^3p_a - 8519.427734ba^5p_bp_a - 1421.62207b^2p_a^6 - 3003.146484b^2p_b^2p_a^4 - 5889.585937b^2p_b^4p_a^2 \\ & - 5470.875b^2p_b^6 - 3525.647461b^2a^2p_a^4 - 12018.02344b^2a^2p_b^2p_a^2 - 24168.72656b^2a^2p_b^4 - 5942.170898b^2a^4p_a^2 \\ & - 20062.8457b^2a^4p_b^2 - 4113.364258b^2a^6 + 315b^3ap_bp_a^3 + 2136.5625b^3ap_b^3p_a + 1465.03125b^3a^3p_bp_a \\ & + 162.2109375b^4p_a^4 + 167.859375b^4p_b^2p_a^2 + 87.1875b^4p_b^4 + 374.765625b^4a^2p_a^2 + 776.578125b^4a^2p_b^2 \\ & + 553.2421875b^4a^4 - 79.3125b^5ap_bp_a + 21.84375b^6p_a^2 - 34.21875b^6a^2 - 56633.10778ap_b^2p_a^6 \\ & - 664522.0057ap_b^4p_a^4 - 2172540.056ap_b^6p_a^2 - 1891469.812ap_b^8 - 1876.875a^3p_a^6 - 145198.4552a^3p_b^2p_a^4 \\ & - 1398715.586a^3p_b^4p_a^2 - 2556386.332a^3p_b^6 - 6418.125a^5p_a^4 - 183727.9422a^5p_b^2p_a^2 - 973061.8752a^5p_b^4 \\ & - 10565.625a^7p_a^2 - 133101.9862a^7p_b^2 - 8199.375a^9 + 53817.79528bp_bp_a^7 + 710798.4675bp_b^3p_a^5 \\ & + 2272182.33bp_b^5p_a^3 + 1839693.516bp_b^7p_a + 160439.7729ba^2p_bp_a^5 + 1427243.907ba^2p_b^3p_a^3 + 2415830.971ba^2p_b^5p_a \\ & + 165225.9908ba^4p_bp_a^3 + 837383.2245ba^4p_b^3p_a + 81215.50312ba^6p_bp_a + 9417.975b^2ap_a^6 + 292971.191b^2ap_b^2p_a^4 \end{aligned}$$

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\begin{aligned}
&+ 988776.6406b^2ap_b^4p_a^2 + 816411.2344b^2ap_b^6 + 45019.95234b^2a^3p_a^4 + 613135.4477b^2a^3p_b^2p_a^2 + 1087849.826b^2a^3p_b^4 \\
&+ 63275.38219b^2a^5p_a^2 + 371449.8691b^2a^5p_b^2 + 34743.59859b^2a^7 - 44962.0375b^3p_bp_a^5 - 196932.9167b^3p_b^3p_a^3 \\
&- 168735.9375b^3p_b^5p_a - 90286.9375b^3a^2p_bp_a^3 - 214383.9062b^3a^2p_b^3p_a - 49381.9375b^3a^4p_bp_a - 5703.75b^4ap_a^4 \\
&- 34414.96875b^4ap_b^2p_a^2 - 28382.34375b^4ap_b^4 - 13341.6875b^4a^3p_a^2 - 42076.71875b^4a^3p_b^2 - 8609.5625b^4a^5 \\
&+ 2691.375b^5p_bp_a^3 + 2611.6875b^5p_b^3p_a + 3555b^5a^2p_bp_a + 748.5b^6ap_a^2 + 371.8125b^6ap_b^2 \\
&+ 586.125b^6a^3 + 1196.507813p_a^{10} - 159673.8032p_b^2p_a^8 - 1641568.942p_b^4p_a^6 - 2960115.624p_b^6p_a^4 \\
&+ 2938677.994p_b^8p_a^2 + 5510914.154p_b^{10} + 5982.539062a^2p_a^8 - 519492.4567a^2p_b^2p_a^6 - 2084770.628a^2p_b^4p_a^4 \\
&+ 9255710.833a^2p_b^6p_a^2 + 22633652.12a^2p_b^8 + 16657.26563a^4p_a^6 + 11580.87035a^4p_b^2p_a^4 + 6320745.164a^4p_b^4p_a^2 \\
&+ 22511170.17a^4p_b^6 + 38681.01562a^6p_a^4 + 1009573.848a^6p_b^2p_a^2 + 7737868.879a^6p_b^4 + 55890.35156a^8p_a^2 \\
&+ 855115.0257a^8p_b^2 + 38365.57031a^{10} + 104208.4468bap_bp_a^7 - 719620.8838bap_b^3p_a^5 - 9442022.033bap_b^5p_a^3 \\
&- 14954112.74bap_b^7p_a - 337348.7184ba^3p_bp_a^5 - 7166949.938ba^3p_b^3p_a^3 - 18969319.83ba^3p_b^5p_a - 979668.9518ba^5p_bp_a^3 \\
&- 6736954.084ba^5p_b^3p_a - 606224.8972ba^7p_bp_a - 67529.7107b^2p_a^8 - 328056.9135b^2p_b^2p_a^6 - 604112.0102b^2p_b^4p_a^4 \\
&- 2056995.902b^2p_b^6p_a^2 - 2072877.961b^2p_b^8 - 244759.8088b^2a^2p_a^6 - 1377475.703b^2a^2p_b^2p_a^4 - 6319227.236b^2a^2p_b^4p_a^2 \\
&- 9765453.703b^2a^2p_b^6 - 486328.8539b^2a^4p_a^4 - 4193265.886b^2a^4p_b^2p_a^2 - 9977382.514b^2a^4p_b^4 - 574300.5219b^2a^6p_a^2 \\
&- 3318739.244b^2a^6p_b^2 - 285360.578b^2a^8 - 48363.24609b^3ap_bp_a^5 + 571034.8389b^3ap_b^3p_a^3 + 1254332.227b^3ap_b^5p_a \\
&+ 263883.0684b^3a^3p_bp_a^3 + 1393103.161b^3a^3p_b^3p_a + 323077.6582b^3a^5p_bp_a + 26491.11182b^4p_a^6 + 53931.81958b^4p_b^2p_a^4 \\
&+ 84375.41602b^4p_b^4p_a^2 + 64111.07812b^4p_b^6 + 83415.59033b^4a^2p_a^4 + 252372.0513b^4a^2p_b^2p_a^2 + 368341.8457b^4a^2p_b^4 \\
&+ 135378.2358b^4a^4p_a^2 + 355166.8528b^4a^4p_b^2 + 85777.04639b^4a^6 - 9617.449219b^5ap_bp_a^3 - 25019.15625b^5ap_b^3p_a \\
&- 19684.86328b^5a^3p_bp_a - 1500.076172b^6p_a^4 - 1483.59375b^6p_b^2p_a^2 - 807.46875b^6p_b^4 - 3822.011719b^6a^2p_a^2 \\
&- 7343.71875b^6a^2p_b^2 - 5661.779297b^6a^4 + 682.96875b^7ap_bp_a - 150.4570312b^8p_a^2 + 267.9492188b^8a^2 \\
&+ 0.000000019ap_a^{10} - 2222750.754ap_b^2p_a^8 - 66857249.91ap_b^4p_a^6 - 538068213.8ap_b^6p_a^4 - 1359984055ap_b^8p_a^2 \\
&- 1003462695ap_b^{10} - 23930.15625a^3p_a^8 - 6514999.713a^3p_b^2p_a^6 - 189457894.4a^3p_b^4p_a^4 - 1122739896a^3p_b^6p_a^2 \\
&- 1497734429a^3p_b^8 - 105105a^5p_a^6 - 9198318.154a^5p_b^2p_a^4 - 215126247.9a^5p_b^4p_a^2 - 672838783.3a^5p_b^6 \\
&- 230245.3125a^7p_a^4 - 10307996.08a^7p_b^2p_a^2 - 106623077.1a^7p_b^4 - 298151.25a^9p_a^2 - 6113345.282a^9p_b^2 \\
&- 183525.7812a^{11} + 2186855.519bp_bp_a^9 + 70689421.19bp_b^3p_a^7 + 567616454.8bp_b^5p_a^5 + 1395505442bp_b^7p_a^3 \\
&+ 985830627bp_b^9p_a + 8700002.955ba^2p_bp_a^7 + 210593006.1ba^2p_b^3p_a^5 + 1154609029ba^2p_b^5p_a^3 + 1461382118ba^2p_b^7p_a \\
&+ 13341807.05ba^4p_bp_a^5 + 229145362.5ba^4p_b^3p_a^3 + 648615207.5ba^4p_b^5p_a + 11914097.29ba^6p_bp_a^3 + 101335229.2ba^6p_b^3p_a \\
&+ 5212222.505ba^8p_bp_a + 334168.0217b^2ap_a^8 + 28876791.16b^2ap_b^2p_a^6 + 253374637.2b^2ap_b^4p_a^4 + 630908168.3b^2ap_b^6p_a^2 \\
&+ 452759758.9b^2ap_b^8 + 2401602.027b^2a^3p_a^6 + 89626833.51b^2a^3p_b^2p_a^4 + 522756983.3b^2a^3p_b^4p_a^2 + 678110783.5b^2a^3p_b^6 \\
&+ 5263424.889b^2a^5p_a^4 + 101819538b^2a^5p_b^2p_a^2 + 300568081b^2a^5p_b^4 + 5454581.78b^2a^7p_a^2 + 46318266.48b^2a^7p_b^2 \\
&+ 2386069.023b^2a^9 - 4771819.608b^3p_bp_a^7 - 54260792.72b^3p_b^3p_a^5 - 150350620.1b^3p_b^5p_a^3 - 109960281.2b^3p_b^7p_a \\
&- 14084153.38b^3a^2p_bp_a^5 - 109875219.8b^3a^2p_b^3p_a^3 - 158622397.7b^3a^2p_b^5p_a - 14362666.38b^3a^4p_bp_a^3 - 59902617.86b^3a^4p_b^3p_a \\
&- 5794003.46b^3a^6p_bp_a - 456890.7891b^4ap_a^6 - 9122240.791b^4ap_b^2p_a^4 - 27289800.63b^4ap_b^4p_a^2 - 20031485.45b^4ap_b^6 \\
&- 1697883.471b^4a^3p_a^4 - 19856348.51b^4a^3p_b^2p_a^2 - 30961182.66b^4a^3p_b^4 - 2173053.832b^4a^5p_a^2 - 11659549.18b^4a^5p_b^2 \\
&- 1043449.895b^4a^7 + 740881.7849b^5p_bp_a^5 + 3012774.733b^5p_b^3p_a^3 + 2361576.094b^5p_b^5p_a + 1633715.997b^5a^2p_bp_a^3 \\
&+ 3625948.363b^5a^2p_b^3p_a + 953235.4062b^5a^4p_bp_a + 75940.125b^6ap_a^4 + 414392.6953b^6ap_b^2p_a^2 + 321831b^6ap_b^4 \\
&+ 188319.6094b^6a^3p_a^2 + 546404.8516b^6a^3p_b^2 + 121787.6719b^6a^5 - 24928.4375b^7p_bp_a^3 - 23933.53125b^7p_b^3p_a \\
&- 35812.125b^7a^2p_bp_a - 6488.25b^8ap_a^2 - 3251.320312b^8ap_b^2 - 5101.171875b^8a^3 + 13261.29492p_a^{12} \\
&- 8145433.975p_b^2p_a^{10} - 197895054.1p_b^4p_a^8 - 1174039780p_b^6p_a^6 - 1411597107p_b^8p_a^4 + 2093095271p_b^{10}p_a^2 \\
&+ 2963833372p_b^{12} + 79567.76953a^2p_a^{10} - 36093324.22a^2p_b^2p_a^8 - 553864892.2a^2p_b^4p_a^6 - 564909062.9a^2p_b^6p_a^4 \\
&+ 8458489476a^2p_b^8p_a^2 + 13893955400a^2p_b^{10} + 258744.8145a^4p_a^8 - 33506279.75a^4p_b^2p_a^6 + 301940153.3a^4p_b^4p_a^4 \\
&+ 6903327778a^4p_b^6p_a^2 + 15961242750a^4p_b^8 + 712391.3672a^6p_a^6 + 28130142.75a^6p_b^2p_a^4 + 1549209851a^6p_b^4p_a^2 \\
&+ 6741451247a^6p_b^6 + 1367766.299a^8p_a^4 + 71440871.35a^8p_b^2p_a^2 + 1000986650a^8p_b^4 + 1601862.926a^{10}p_a^2 \\
&+ 42972167.82a^{10}p_b^2 + 893844.5293a^{12} + 4104708.453bap_bp_a^9 - 24773301.85bap_b^3p_a^7 - 1512129012bap_b^5p_a^5 \\
&- 7982165871bap_b^7p_a^3 - 9403407688bap_b^9p_a - 18735933.67ba^3p_bp_a^7 - 926132644.9ba^3p_b^3p_a^5 - 7629657192ba^3p_b^5p_a^3 \\
&- 13810769200ba^3p_b^7p_a - 80369405.75ba^5p_bp_a^5 - 1809025391ba^5p_b^3p_a^3 - 6337537930ba^5p_b^5p_a - 96022382.9ba^7p_bp_a^3 \\
&- 977625051.4ba^7p_b^3p_a - 42155337.1ba^9p_bp_a - 3010397.046b^2p_a^{10} - 38271790.12b^2p_b^2p_a^8 - 69004738.74b^2p_b^4p_a^6 \\
&- 275435325.2b^2p_b^6p_a^4 - 1306950815b^2p_b^8p_a^2 - 1230754804b^2p_b^{10} - 14094602.09b^2a^2p_a^8 - 172435655.8b^2a^2p_b^2p_a^6 \\
&- 1113225333b^2a^2p_b^4p_a^4 - 4999899699b^2a^2p_b^6p_a^2 - 6289744221b^2a^2p_b^8 - 34378874.51b^2a^4p_a^6 - 644952307.4b^2a^4p_b^2p_a^4 \\
&- 4091219148b^2a^4p_b^4p_a^2 - 7421673472b^2a^4p_b^6 - 54921751.52b^2a^6p_a^4 - 953761771.3b^2a^6p_b^2p_a^2 - 3174339126b^2a^6p_b^4 \\
&- 50194509.7b^2a^8p_a^2 - 474377384.9b^2a^8p_b^2 - 19919401.83b^2a^{10} - 9727701.031b^3ap_bp_a^7 + 68812106.94b^3ap_b^3p_a^5 \\
&+ 718349714.7b^3ap_b^5p_a^3 + 979748383.9b^3ap_b^7p_a + 29070040.01b^3a^3p_bp_a^5 + 576483756.8b^3a^3p_b^3p_a^3 + 1348435577b^3a^3p_b^5p_a \\
&+ 86282930.86b^3a^5p_bp_a^3 + 522828208.6b^3a^5p_b^3p_a + 50342565.51b^3a^7p_bp_a + 2748637.477b^4p_a^8 + 15259882.88b^4p_b^2p_a^6 \\
&+ 28992453.64b^4p_b^4p_a^4 + 61968321.64b^4p_b^6p_a^2 + 50575935.46b^4p_b^8 + 10905214b^4a^2p_a^6 + 68865786.4b^4a^2p_b^2p_a^4 \\
&+ 233296505.2b^4a^2p_b^4p_a^2 + 285309521.9b^4a^2p_b^6 + 22450902.58b^4a^4p_a^4 + 167918842.1b^4a^4p_b^2p_a^2 + 329241858.8b^4a^4p_b^4 \\
&+ 24852043.44b^4a^6p_a^2 + 122057814.3b^4a^6p_b^2 + 11113157.45b^4a^8 - 531266.1687b^5ap_bp_a^5 - 14417141.78b^5ap_b^3p_a^3 \\
&- 22578238.97b^5ap_b^5p_a - 6812088.95b^5a^3p_bp_a^3 - 27759004.41b^5a^3p_b^3p_a - 7054021.907b^5a^5p_bp_a - 382659.1b^6p_a^6 \\
&- 752867.4639b^6p_b^2p_a^4 - 1066061.153b^6p_b^4p_a^2 - 757074.6562b^6p_b^6 - 1410039.876b^6a^2p_a^4 - 3955667.832b^6a^2p_b^2p_a^2 \\
&- 4931773.655b^6a^2p_b^4 - 2296169.313b^6a^4p_a^2 - 5205081.391b^6a^4p_b^2 - 1414416.26b^6a^6 + 147926.0742b^7ap_bp_a^3 \\
&+ 280261.3359b^7ap_b^3p_a + 244195.0547b^7a^3p_bp_a + 14067.47461b^8p_a^4 + 13530.41895b^8p_b^2p_a^2 + 7454.917969b^8p_b^4 \\
&+ 37930.64062b^8a^2p_a^2 + 70186.94824b^8a^2p_b^2 + 58166.45508b^8a^4 - 6036.878906b^9ap_bp_a + 1108.072266b^{10}p_a^2 \\
&- 2184.275391b^{10}a^2 + \mathcal{O}(13)
\end{aligned}
$$

The most impressive simple structure of this tiny little formula and its evident meaningfulness immediately show the superiority of our simplification algorithm.